Harmonic oscillator realization of the canonical q-transformation

1991 J. Phys. A: Math. Gen. 24 L63

(http://iopscience.iop.org/0305-4470/24/2/003)

J. Phys. A: Math. Gen. 24 (1991) L63–L68. Printed in the UK

LETTER TO THE EDITOR

Harmonic oscillator realization of the canonical q-transformation

A T Filippov, Debashis Gangopadhyay and A P Isaev

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow, USSR

Received 31 October 1990

Abstract. Exact realization of the canonical q-transformation for q-oscillators is obtained in the context of the harmonic oscillator realization of q-oscillators.

Quantum Lie algebras first appeared in investigations of the quantum inverse scattering problem during the study of the Yang–Baxter equations [1]. They can be considered as some 'deformation' of the Lie algebra with the deformation parameter 's' or $q = e^s$, such that the usual Lie algebra is reproduced in the limit $s \to 0$, i.e. $q \to 1$. It has been pointed out by Drinfeld [2] that these deformed structures are essentially connected with quasi-triangular Hopf algebras, and the generalization to all simple Lie algebras has been given [2, 3]. There are versions of deformed Kac–Moody and Virasoro algebras [4], the realization of quantum $SU(2)_q$ algebra in terms of q-oscillators has been extensively studied [5], and there exist q-oscillator realizations of many other quantum algebras [6]. In the context of the harmonic oscillator realization of q-oscillators [7], it has been shown in [8] that the general solution to this realization contains two arbitrary functions of q. The known realization results when these functions are taken to be unity. In this letter we establish a new harmonic oscillator realization of bosonic q-oscillators which can be interpreted as canonical q-transformations. We obtain exact expressions for the transformation coefficients and again demonstrate the existence of arbitrary functions of q which, in the limit $q \to 1$, are related to the parameters of the $SL(2, R)$ group. We also briefly discuss the features that distinguish our transformations from the transformations of the $SL(2, R)_q$ group.

The equations characterizing the q-deformed bosonic oscillator system are

$$\tilde{a}\tilde{a}^+ - q\tilde{a}^+\tilde{a} = q^{-N} \qquad N^+ = N \tag{1}$$

$$N\tilde{a} = \tilde{a}(N-1) \qquad N\tilde{a}^+ = \tilde{a}^+(N+1) \tag{2}$$

$\tilde{a}$, $\tilde{a}^+$ and N are the annihilation, creation and number operators respectively. Usual harmonic oscillators $\hat{a}$, $\hat{a}^+$ are described by

$$\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1 \qquad \hat{N} = \hat{a}^+\hat{a} \tag{3}$$

$$\hat{N}\hat{a} = \hat{a}(\hat{N}-1) \qquad \hat{N}\hat{a}^+ = \hat{a}^+(\hat{N}+1) \tag{4}$$

where $\hat{N}$ is the number operator. According to [8] the most general harmonic oscillator realization of the q-oscillator represented in the simplest form

$$\tilde{a} = \hat{a}u(\hat{N}) \qquad \tilde{a}^+ = u(\hat{N})\hat{a}^+ \tag{5}$$

is

$$\tilde{a} = \hat{a}\left(\frac{q^{\hat{N}}\phi_1 - q^{-\hat{N}}\phi_2}{\hat{N}(q-q^{-1})}\right)^{1/2} \qquad \tilde{a}^+ = \left(\frac{q^{\hat{N}}\phi_1 - q^{-\hat{N}}\phi_2}{\hat{N}(q-q^{-1})}\right)^{1/2}\hat{a}^+$$
$$N = \hat{N} - (1/s)\ln\phi_2 \tag{6}$$

where the functions $\phi_1(q, \hat{N})$ and $\phi_2(q, \hat{N})$ are such that $\phi_i(q, \hat{N}+1) = \phi_i(q, \hat{N})$ and belong to what we name as class P of the periodic functions. Considering the feature $[\phi_i, \hat{a}] = [\phi_i, \hat{a}^+] = [\phi_i, \hat{N}] = 0$, we can take ϕ_i (without loss of generality) to be functions of q only. Choosing $\phi_1 = \phi_2 = 1$ gives the well known realization [7].

We now seek the representation for q-oscillators in terms of usual harmonic oscillators in the form

$$a = \hat{a}u(\hat{N}) + v(\hat{N})\hat{a}^+ \qquad a^+ = u^*(\hat{N})\hat{a}^+ + \hat{a}v^*(\hat{N}). \tag{7}$$

For future convenience we denote the q-oscillators by a instead of $\tilde{a}$. $u(\hat{N})$, $v(\hat{N})$ are functions to be determined subsequently.

For simplicity, we choose the fundamental q-commutator as

$$aa^+ - q^2a^+a = 1. \tag{8}$$

This is equivalent to (1) under the identification

$$a = q^{N/2}\tilde{a} \qquad a^+ = \tilde{a}^+q^{N/2}. \tag{9}$$

Equation (2) becomes

$$Na = a(N-1) \qquad Na^+ = a^+(N+1). \tag{10}$$

Using (6) and (9) we can write (7) in the form of the canonical q-transformation ('Bogolubov q-transformation')

$$a' = a\tilde{u}(\hat{N}) + \tilde{v}(\hat{N})a^+$$
$$a'^+ = \tilde{u}^*(\hat{N})a^+ + a\tilde{v}^*(\hat{N}) \tag{11a}$$

i.e.

$$\begin{pmatrix} a' \\ a'^+ \end{pmatrix} = \begin{pmatrix} \tilde{u}(\hat{N}+1) & \tilde{v}(\hat{N}) \\ \tilde{v}^*(\hat{N}+1) & \tilde{u}^*(\hat{N}) \end{pmatrix}\begin{pmatrix} a \\ a^+ \end{pmatrix} \tag{11b}$$

where (a, a^+), (a', a'^+) satisfy (8), $*$ denotes complex conjugation and

$$\tilde{u}(\hat{N}) = \left(\frac{\hat{N}(q^2-1)}{\phi_1q^{2\hat{N}} - \phi_2}\right)^{1/2} u(\hat{N}) \qquad \tilde{v}(\hat{N}) = v(\hat{N})\left(\frac{\hat{N}(q^2-1)}{\phi_1q^{2\hat{N}} - \phi_2}\right)^{1/2}. \tag{12}$$

Thus our transformations (11) act on the two-dimensional quantum space of vectors (a, a^+) satisfying (8) and preserve this property for (a', a'^+), so we can interpret our transformation (11b) as an element of the q-deformed SL(2, R) group. However, this q-deformed group is not related to the quantum group SL$(2, R)_q$ as the quantities $\tilde{u}$, $\tilde{v}$, u^*, v^* in (11) are commuting operators while the elements of the SL$(2, R)_q$ matrix U have non-trivial commutation relations. Indeed, in the theory of quantum groups [2, 3] SL$(2, R)_q$ transformations have the form similar to (11b):

$$\begin{pmatrix} a' \\ a'^+ \end{pmatrix} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}\begin{pmatrix} a \\ a^+ \end{pmatrix} = U\begin{pmatrix} a \\ a^+ \end{pmatrix}. \tag{13a}$$

However, for the $SL(2, R)_q$ case we suggest that u, v, u^*, v^* are mutually non-commutative objects but they commute with (a, a^+). The conjugate transformations to (13a) are:

$$(a'', a''^+) = (a, a^+)\begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}. \tag{13b}$$

Then the condition that (a, a^+), (a', a'^+) and (a'', a''^+) satisfy (8) (i.e. (13a, b) are canonical q-transformations) gives us the condition $\det_{q^2}(U) = uu^* - q^2 v^* v = 1$ and the braiding rules for the elements of the matrix U [see 3 and references therein]:

$$uv^* = q^2 v^* u \tag{14a}$$

$$vu^* = q^2 u^* v \tag{14b}$$

$$vv^*(q^{-2} - q^2) = u^* u - uu^* \tag{14c}$$

$$uv = q^2 vu \qquad v^* u^* = q^2 u^* v^* \tag{14d}$$

$$v^* v = vv^*. \tag{14e}$$

In this letter we shall concentrate on the canonical q-transformations (11). We wish to determine $u(\hat{N})$ and $v(\hat{N})$ of the representation (7). Substituting (7) in (8) and using (3) and (4) we have

$$F(\hat{N}+1) - q^2 F(\hat{N}) + G(\hat{N}) - q^2 G(\hat{N}+1) = 1 \tag{15}$$

$$u(\hat{N}) v^*(\hat{N}+1) = q^2 v^*(\hat{N}) u(\hat{N}+1) \tag{16a}$$

$$u^*(\hat{N}) v(\hat{N}+1) = q^2 v(\hat{N}) u^*(\hat{N}+1) \tag{16b}$$

where

$$F(\hat{N}) = \hat{N} u^*(\hat{N}) u(\hat{N}) \qquad G(\hat{N}) = \hat{N} v^*(\hat{N}) v(\hat{N}). \tag{16c}$$

It is interesting to note that (16a, b) can be written in the form (14a, b) under the convention that in the product of two operators the operator which is a function of $\hat{N}$ is placed to the left of the operator which is a function of $(\hat{N}+1)$.

Multiplying (16a) and (16b) gives

$$\frac{G(\hat{N}+1)}{F(\hat{N}+1)} = q^4 \frac{G(\hat{N})}{F(\hat{N})}$$

whose solution is

$$\frac{G(\hat{N})}{F(\hat{N})} = q^{4\hat{N}} W(q, \hat{N}) \tag{17}$$

where the arbitrary function of q, $W(q, \hat{N}) = W(q, \hat{N}+1)$ and is thus some arbitrary P-function. For reasons discussed before, W may be taken as a function of q only. Substituting (17) in (15) we get

$$F(\hat{N}+1)\{1 - q^{4(\hat{N}+2)} \tilde{W}\} - q^2 F(\hat{N})\{1 - q^{4\hat{N}} \tilde{W}\} = 1 \tag{18}$$

where $W = q^2 \tilde{W}$. To solve the functional equation (18) we first determine the solution $F_0(\hat{N})$ to the corresponding homogeneous equation

$$F_0(\hat{N}+1)\{1 - q^{4(\hat{N}+2)} \tilde{W}\} = q^2 F_0(\hat{N})\{1 - q^{4\hat{N}} \tilde{W}\}. \tag{19}$$

The solution of (19) has the form (see appendix)

$$F_0(\hat{N}) = Q(\hat{N}) Q(\hat{N}+1) P(\hat{N}, q) \tag{20}$$

where

$$Q(\hat{N}) = \frac{q^{\hat{N}}}{1 - q^{4\hat{N}}\tilde{W}} \tag{21}$$

and $P(\hat{N}, q)$ is an arbitrary P-function. We shall soon see that for a suitable choice of initial conditions the general solution of (18) is independent of $P(\hat{N}, q)$.

We represent the general solution of (18) as

$$F(\hat{N}) = F_0(\hat{N})\, Y(\hat{N}) \tag{22}$$

where $Y(\hat{N})$ is to be determined. Putting (22) in (18) yields

$$Y(\hat{N}+1, q) = Y(\hat{N}, q) + \frac{(1 - Wq^{4\hat{N}+2})}{q^{2\hat{N}+3}P}. \tag{23}$$

With the use of standard techniques the solution to (23) is found to be

$$Y(\hat{N}, q) = Y(0, q) + \frac{q^{-\hat{N}} - Wq^{\hat{N}}}{q^2 P(\hat{N}, q)}[\hat{N}] \tag{24}$$

where $[x] = q^x - q^{-x}/(q - q^{-1})$ and $Y(0, q)$ is some initial value. Using (24), (22) and (16c) we see that the condition $u(\hat{N})|_{\hat{N}=0} < \infty$ leads to $Y(0, q) = 0$. Using (20) and (22) we thus arrive at our solutions for $F(\hat{N})$ and $G(\hat{N})$

$$F(\hat{N}) = \frac{q^{\hat{N}-1}(1 - q^{2\hat{N}}W)}{(1 - q^{4\hat{N}-2}W)(1 - q^{4\hat{N}+2}W)}[\hat{N}]$$
$$G(\hat{N}) = q^{4\hat{N}}F(\hat{N})\,W. \tag{25}$$

This is independent of P as promised, but the dependence on W is non-trivial as we shall see. One can verify that the solutions (25) do indeed satisfy (18). Using definitions (16c) we have

$$u(\hat{N}) = |u(\hat{N})|\, e^{i\alpha(q,\hat{N})} \qquad v(\hat{N}) = q^{2\hat{N}}W^{1/2}|u(\hat{N})|\, e^{i\beta(q,\hat{N})}$$

with

$$|u(\hat{N})| = \left(\frac{q^{\hat{N}-1}\{1 - q^{2\hat{N}}W\}}{\{1 - q^{4\hat{N}-2}W\}\{1 - q^{4\hat{N}+2}W\}}\,\frac{[\hat{N}]}{\hat{N}}\right)^{1/2} \tag{26}$$

and $\alpha(q, \hat{N})$, $\beta(q, \hat{N})$ some arbitrary phase factors. Equations (16a, b) are also trivially satisfied by (26). Therefore the representation (7) for q-oscillators, as realized in terms of ordinary oscillators, may be written as

$$a = \hat{a}\{|u(\hat{N})|\, e^{i\alpha}\} + \{q^{2\hat{N}}W^{1/2}|u(\hat{N})|\, e^{i\beta}\}\hat{a}^+$$
$$a^+ = \{|u(\hat{N})|\, e^{-i\alpha}\}\hat{a}^+ + \hat{a}\{q^{2\hat{N}}W^{1/2}|u(\hat{N})|\, e^{-i\beta}\}. \tag{27a}$$

Using (26), (27a) and (12) (for $\phi_1 = \phi_2 = 1$) we can obtain the canonical q-transformation (11) in the form:

$$a' = a\{|\bar{u}(\hat{N})|\, e^{i\alpha}\} + \{q^{2\hat{N}}W^{1/2}|\bar{u}(\hat{N})|\, e^{i\beta}\}a^+$$
$$a'^+ = \{|\bar{u}(\hat{N})|\, e^{-i\alpha}\}a^+ + a\{q^{2\hat{N}}W^{1/2}|\bar{u}(\hat{N})|\, e^{-i\beta}\} \tag{27b}$$

with

$$|\bar{u}(\hat{N})| = \left(\frac{1 - q^{2\hat{N}}W}{\{1 - q^{4\hat{N}-2}W\}\{1 - q^{4\hat{N}+2}W\}}\right).$$

We now comment on the presence of the function $W(q)$. Let $q \to 1$ and α, β be independent of $\hat{N}$ in (26). Then we obtain

$$u(\hat{N}) = \left(\frac{1}{1-W(1)}\right)^{1/2} e^{i\alpha(1)} \qquad v(\hat{N}) = \left(\frac{W(1)}{1-W(1)}\right)^{1/2} e^{i\beta(1)}. \tag{28}$$

Thus, for $q = 1$, substituting (28) in (11) we get the usual SL(2, R) canonical transformation of the ordinary oscillators where $\alpha(1)$, $\beta(1)$ and $W(1)$ are the parameters of the SL(2, R) transformations. Therefore, $\alpha(q, \hat{N})$, $\beta(q, \hat{N})$ and $W(q)$ are parameters of the q-deformed SL(2, R) transformations (11). Note that for $W = 0$ (and $\alpha = 0$, $\beta = 0$) representation (7) coincides with (6) when $\phi_1 = \phi_2 = 1$ ($Y(q, 0) = 0$). We can obtain the case $\phi_1 \neq 1$ and $\phi_2 \neq 1$ if we consider the situation when $Y(q, 0) \neq 0$ in (24).

Finally, let us write down an expression for the number operator using the general solution (6). We have

$$\tilde{a}^+ \tilde{a} = \frac{\phi_1 \phi_2 q^N - q^{-N}}{q - q^{-1}}. \tag{29}$$

Then, using (9)

$$a^+ a = \frac{\phi_1 \phi_2 q^{2\hat{N}} - 1}{q^2 - 1} \tag{30}$$

so that

$$N = \frac{1}{2s} \ln\{\phi + \phi(q^2 - 1) a^+ a\} \tag{31}$$

where $\phi = \{\phi_1 \phi_2\}^{-1}$ and a^+, a satisfy (8), with their harmonic oscillator realizations given by (27a). Equations (10) are also satisfied by N as defined in (31). From expression (31) for N we see that the spectrum is non-trivially shifted. One can, using (31), write out an explicit relation for N in terms of $\hat{N}$ and this will contain off-diagonal terms. Hence, the spectrum for q-oscillators is modified with respect to that of the usual case defined by $\hat{N}$. This is an interesting point. Another interesting question is: what is the relation between our q-deformed SL(2, R) transformations (11) and the quantum group $SL_q(2, R)$? Whether physical applications of our results are possible is yet another avenue worth pursuing.

Appendix

Equation (19) is of the general form

$$F_0(\hat{N}+1)\tilde{Q}(\hat{N}+k) = g(q) F_0(\hat{N}) \tilde{Q}(\hat{N}) \tag{A1}$$

where $g(q)$ and $\tilde{Q}(\hat{N}+k)$, $(k = 1, 2, \ldots)$, are known functions while $F_0(\hat{N})$ is the function to be determined. In (19)

$$k = 2 \qquad g(q) = q^2 \qquad \text{and} \qquad \tilde{Q}(\hat{N}) = \{1 - q^{4\hat{N}} \tilde{W}\}. \tag{A2}$$

Note that if F_1, F_2 are solutions of (A1) then

$$\frac{F_1(\hat{N}+1)}{F_2(\hat{N}+1)} = \frac{F_1(\hat{N})}{F_2(\hat{N})} = P(\hat{N}, q) \tag{A3}$$

and thus $P(\hat{N}, q)$ is a P-function. It means that the general solution of (A1) is a product of a special solution and an arbitrary P-function. We can search for the general solution of (A1) in the form:

$$F_0(\hat{N}) = \left\{ \prod_{i=0}^{k-1} Q(\hat{N}+i) \right\} P(\hat{N}, q). \tag{A4}$$

Putting (A4) in (A1) gives

$$Q(\hat{N}+k)\bar{Q}(\hat{N}+k) = g(q)Q(\hat{N})\bar{Q}(\hat{N})$$

which after simplification results in

$$Q(\hat{N}) = \frac{g(q)^{\hat{N}/k}}{\bar{Q}(\hat{N})} \tilde{P}(\hat{N}, q) \tag{A5}$$

where $\tilde{P}(\hat{N}+k, q) = \tilde{P}(\hat{N}, q)$ is an arbitrary periodic function and we can put this function to unity without loss of generality ($\Pi_{i=0}^{k-1} \tilde{P}(\hat{N}, q)$ is a P-function). Now using (A2), (A4) and (A5) one arrives at solutions (20) and (21).

References

[1] Faddeev L D 1982 *Les Houches Lectures* (Amsterdam: North-Holland)
Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics* **151** (Berlin: Springer) p 61
Kulish P P and Reshitikhin N Yu 1983 *Zap. Nauk. Semin. LOMI* **101** 101; 1983 *J. Sov. Math.* **23** 2435
Sklyanin E K 1982 *Funk. Anal. Appl.* **16** 27; 1983 *Funk. Anal. Appl.* **17** 34

[2] Drinfeld V G 1986 *Proc. Int. Congress of Mathematics, Berkely* vol 1 (Providence, RI: American Mathematical Society) p 798
Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247; 1987 *Commun. Math. Phys.* **102** 537
Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613
Manin Y I 1988 *Quantum Groups and Non-Commutative Geometry* (Montreal: Centre de Recherches Mathematiques)
Rosso M 1989 *Commun. Math. Phys.* **124** 307
Arnaudon D and Roche P 1989 *Lett. Math. Phys.* **17** 295

[3] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1989 *Algebra and Analysis* **1** 178

[4] Bernard D 1989 *Lett. Math. Phys.* **17** 239
Frenkel I B and Jing N 1988 *Proc. Natl Acad. Sci. USA* **8S** 9373
Polychronakos A P 1990 Consistency conditions and representations of a q-deformed Virasoro algebra *Preprint* University of Florida UFIFT-HEP-90-14
Chaichian M, Ellinas D and Popowicz Z 1990 Quantum conformal algebra with central extension *Preprint* University of Helsinki HU-TFT-90-37

[5] MacFarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Sen C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983

[6] Chaichian M and Kulish P 1990 *Phys. Lett.* **234B** 72
Chaichian M, Kulish P and Lukierski J 1990 *Phys. Lett.* **237B** 401
Kulish P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415
Floreanini R, Spiridonov V P and Vinet L 1990 *Phys. Lett.* **242B** 383
Floreanini R, Spiridonov V P and Vinet L 1990 q-oscillator realisations of the quantum superalgebras $\mathrm{sl}_q(m, n)$ and $\mathrm{osp}_q(m, 2n)$ *Preprint* UCLA-90-TEP21

[7] Jannussis A, Brodimas G, Sourlas D and Zisis V 1981 *Lett. Nuovo Cimento* **30** 123
Polychronakos A P 1989 A classical realisation of quantum algebras *Preprint* University of Florida HEP-89-23

[8] Gangopadhyay D and Isaev A P 1990 *JINR Rapid Commun.* **43** in press